

# Character Sums and Congruences with $n!$

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## Abstract

We estimate character sums with  $n!$ , on average, and individually. These bounds are used to derive new results about various congruences modulo a prime  $p$  and obtain new information about the spacings between quadratic nonresidues modulo  $p$ . In particular, we show that there exists a positive integer  $n \ll p^{1/2+\varepsilon}$ , such that  $n!$  is a primitive root modulo  $p$ . We also show that every nonzero congruence class  $a \not\equiv 0 \pmod{p}$  can be represented as a product of 7 factorials,  $a \equiv n_1! \dots n_7! \pmod{p}$ , where  $\max\{n_i \mid i = 1, \dots, 7\} = O(p^{11/12+\varepsilon})$ , and we find the asymptotic formula for the number of such representations. Finally, we show that products of 4 factorials  $n_1!n_2!n_3!n_4!$ , with  $\max\{n_1, n_2, n_3, n_4\} = O(p^{6/7+\varepsilon})$  represent “almost all” residue classes modulo  $p$ , and that products of 3 factorials  $n_1!n_2!n_3!$  with  $\max\{n_1, n_2, n_3\} = O(p^{5/6+\varepsilon})$  are uniformly distributed modulo  $p$ .

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## 1 Introduction

Throughout this paper,  $p$  is an odd prime. Very little seems to be known about the distribution of  $n!$  modulo  $p$ . In **F11** in [7], it is conjectured that about  $p/e$  of the residue classes  $a \pmod{p}$  are missed by the sequence  $n!$ . If this were so, the sequence  $n!$  modulo  $p$  should assume about  $(1 - 1/e)p$  distinct values. Some results in this spirit appear in [2]. The above conjecture immediately implies that every residue class  $a$  modulo  $p$  can be represented as a product of at most two factorials. Unconditionally, it is easy to see that three factorials suffice. Indeed,  $0 \equiv p! \pmod{p}$ , and, as it has been remarked in [5], equation (5), see also [2], the *Wilson theorem* implies that

$$b! \cdot (p - 1 - b)! \equiv (-1)^{b+1} \pmod{p} \quad (1)$$

holds for any  $b \in \{1, \dots, p - 1\}$ . Therefore, if  $a \in \{1, \dots, p - 1\}$ , then with  $b \equiv a^{-1} \pmod{p}$ , we have

$$a \equiv ((p - 1)!)^{r_b} (b - 1)! \cdot (p - 1 - b)! \pmod{p}, \quad (2)$$

where  $r_b \in \{0, 1\}$  is such that  $r_b \equiv b + 1 \pmod{2}$ . However, the above argument does not apply to proving the existence of representations involving factorials of integers of restricted size, neither can it be used for estimation of the number of representations.

In this paper, we first estimate character sums with  $n!$  on the average, and individually. We use these estimates to show that for every  $\varepsilon$  and  $p$  sufficiently large, there exists a value of  $n$  with  $n = O(p^{1/2+\varepsilon})$  and such that  $n!$  is a primitive root modulo  $p$ .

We apply these estimates to prove that every residue class  $a \not\equiv 0 \pmod{p}$ , can be represented as a product of 7 factorials,  $a \equiv n_1! \dots n_7! \pmod{p}$  with  $\max\{n_i \mid i = 1, \dots, 7\} \ll p^{11/12+\varepsilon}$ . If we only want that “most” of the residue classes modulo  $p$  be represented as a product of factorials in the same range as above (and even a slightly better one), then we show that four factorials suffice. Moreover, our results imply that for every  $\varepsilon > 0$  and sufficiently large  $p$ , every residue class  $a \not\equiv 0 \pmod{p}$  can be represented as a product of  $\ell = \lfloor \varepsilon^{-1} \rfloor + 5$  factorials,  $a \equiv n_1! \dots n_\ell! \pmod{p}$ , where  $\max\{n_i \mid i =$

$1, \dots, \ell\} \ll p^{1/2+\varepsilon}$ . We also show that products of three factorials  $n_1!n_2!n_3!$ , with  $\max\{n_1, n_2, n_3\} = O(p^{5/6+\varepsilon})$ , are uniformly distributed modulo  $p$ .

Our basic tools are the *Weil bound* for character sums, see [12, 13, 18], and the *Lagrange theorem* bounding the number of zeros of a non-zero polynomial over a field.

Some of the results of this paper have found applications to the study of arithmetic properties of expressions of the form  $n! + f(n)$ , where  $f(n)$  is a polynomial with integer coefficients (see [14]), or a linearly recurrent sequence of integers (see [15]). In particular, an improvement of a result of Erdős and Stewart [5], obtained in [14], is based on these results.

Throughout the paper the implied constants in symbols ‘ $O$ ’ and ‘ $\ll$ ’ may occasionally, where obvious, depend on integer parameters  $\ell$  and  $d$  and a small real parameter  $\varepsilon > 0$ , and are absolute otherwise (we recall that  $A \ll B$  is equivalent to  $A = O(B)$ ).

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## 2 Character Sums

Let  $\mathbb{F}_p$  be a finite field of  $p$  elements. We always assume that  $\mathbb{F}_p$  is represented by the elements of the set  $\{0, 1, \dots, p-1\}$ .

Let  $\mathcal{X}$  denote the set of multiplicative characters of the multiplicative group  $\mathbb{F}_p^*$  and let  $\mathcal{X}^* = \mathcal{X} \setminus \{\chi_0\}$  be the set of nonprincipal characters.

We also define

$$\mathbf{e}(z) = \exp(2\pi iz/p),$$

which is an additive character of  $\mathbb{F}_p$ .

It is useful to recall the identities

$$\sum_{\chi \in \mathcal{X}} \chi(u) = \begin{cases} 0, & \text{if } u \not\equiv 1 \pmod{p}, \\ p-1, & \text{if } u \equiv 1 \pmod{p}, \end{cases}$$

and

$$\sum_{a=0}^{p-1} \mathbf{e}(au) = \begin{cases} 0, & \text{if } u \not\equiv 0 \pmod{p}, \\ p, & \text{if } u \equiv 0 \pmod{p}, \end{cases}$$

which we will repeatedly use, in particular to relate the number of solutions of various congruences and character sums.

Given  $\chi \in \mathcal{X}$ , a polynomial  $f \in \mathbb{F}_p[X]$ , and an element  $a \in \mathbb{F}_p$ , we consider character sums

$$T(\chi, f, H, N) = \sum_{n=H+1}^{H+N} \chi(n!) \mathbf{e}(f(n))$$

where we simply write  $T(\chi, H, N)$  if  $f$  is identical to zero, and

$$S(a, H, N) = \sum_{n=H+1}^{H+N} \mathbf{e}(an!).$$

We obtain a nontrivial upper bound for “individual” sums  $T(\chi, f, H, N)$ , and also nontrivial upper bounds for the moments of  $T(\chi, f, H, N)$  and  $S(a, H, N)$ .

**Theorem 1.** *Let  $H$  and  $N$  be integers with  $0 \leq H < H + N < p$ . Then for any fixed integer  $d \geq 0$ , the following bound holds:*

$$\max_{\deg f=d} \max_{\chi \in \mathcal{X}^*} |T(\chi, f, H, N)| \ll N^{3/4} p^{1/8} (\log p)^{1/4}.$$

*Proof.* For any integer  $k \geq 0$  we have

$$T(\chi, f, H, N) = \sum_{n=H+1}^{H+N} \chi((n+k)!) \mathbf{e}(f(n+k)) + O(k).$$

Therefore, for any integer  $K \geq 0$ ,

$$T(\chi, f, H, N) = \frac{1}{K} W + O(K), \tag{3}$$

where

$$\begin{aligned} W &= \sum_{k=0}^{K-1} \sum_{n=H+1}^{H+N} \chi((n+k)!) \mathbf{e}(f(n+k)) \\ &= \sum_{n=H+1}^{H+N} \sum_{k=1}^K \chi \left( n! \prod_{i=1}^k (n+i) \right) \mathbf{e}(f(n+k)) \\ &= \sum_{n=H+1}^{H+N} \chi(n!) \sum_{k=0}^{K-1} \chi \left( \prod_{i=1}^k (n+i) \right) \mathbf{e}(f(n+k)). \end{aligned}$$

Applying the Cauchy inequality, we derive

$$\begin{aligned}
|W|^2 &\leq N \sum_{n=H+1}^{H+N} \left| \sum_{k=0}^{K-1} \chi \left( \prod_{i=1}^k (n+i) \right) \mathbf{e}(f(n+k)) \right|^2 \\
&= N \sum_{k,m=0}^{K-1} \sum_{n=H+1}^{H+N} {}^* \chi(\Psi_{k,m}(n)) \mathbf{e}(f(n+k) - f(n+m)),
\end{aligned}$$

where

$$\Psi_{k,m}(X) = \prod_{i=1}^k (X+i) \prod_{j=1}^m (X+j)^{-1}$$

and hereafter  $\sum^*$  means that the poles of the corresponding rational function are excluded from the summation (we also recall that  $|z|^2 = z\bar{z}$  for any complex number  $z$ , and that  $\bar{\chi}(a) = \chi(a^{-1})$  holds for every integer  $a \not\equiv 0 \pmod{p}$  where  $\bar{\chi}$  is the conjugate character of  $\chi$ ).

Clearly, if  $K < p$  then, unless  $k = m$ , the rational function  $\Psi_{k,m}(X)$ , has at least one single root or pole, and thus is not a power of any other rational function modulo  $p$ .

For the  $O(K)$  choices of  $0 \leq k = m \leq K-1$ , we estimate the sum over  $n$  trivially as  $N$ .

For the other  $O(K^2)$  choices of  $0 \leq k, m \leq K-1$ , using the Weil bound, given in Example 12 of Appendix 5 of [18] (see also Theorem 3 of Chapter 6 in [12], or Theorem 5.41 and the comments to Chapter 5 of [13]), we see that, because  $\chi \in \mathcal{X}^*$ , then, for any  $a \in \mathbb{F}_p$ , we have

$$\sum_{n=0}^{p-1} {}^* \chi(\Psi_{k,m}(n)) \mathbf{e}(f(n+k) - f(n+m) + an) = O(Kp^{1/2}).$$

Therefore, by the standard reduction of incomplete sums to complete ones, (see [1]), we deduce

$$\sum_{n=H+1}^{H+N} {}^* \chi(\Psi_{k,m}(n)) \mathbf{e}(f(n+k) - f(n+m)) = O(Kp^{1/2} \log p).$$

Putting everything together, we get

$$W^2 \ll N (KN + K^3 p^{1/2} \log p).$$

Therefore, by (3), we derive

$$T(\chi, f, H, N) \ll NK^{-1/2} + K^{1/2}N^{1/2}p^{1/4}(\log p)^{1/2} + K.$$

Taking  $K = \lfloor N^{1/2}p^{-1/4}(\log p)^{-1/2} \rfloor$ , we finish the proof.  $\square$

It is clear that for any  $\varepsilon > 0$  there exists some  $\delta > 0$ , such that if  $N \geq p^{1/2+\varepsilon}$  then

$$|T(\chi, f, H, N)| \leq Np^{-\delta}.$$

provided that  $p$  is large enough.

Clearly, Theorem 1 immediately implies that among the values of  $n!$ , where  $n = H+1, \dots, H+N$ , there are  $N/2 + O(N^{3/4}p^{1/8}(\log p)^{1/4})$  quadratic residues and nonresidues. Remarking that each change in the value of the Legendre symbol  $(n!/p)$  corresponds to a quadratic non-residue  $n$  we can derive a certain result about the distribution of spacings between quadratic non-residues modulo  $n$  which does not seem to follow from any of the previously known results, see [11].

Let  $n_j$  be the  $j$ th quadratic nonresidue modulo  $p$  and let  $d_j = n_j - n_{j-1}$ , the  $j$ th spacing,  $j = 1, \dots, (p-1)/2$ , where we put  $n_0 = 0$ .

**Corollary 2.** *Let  $J$  be an integer with  $p^{1/2} \log p \leq J \leq (p-1)/2$ . Then the following bound holds:*

$$\sum_{j=1}^J (-1)^j d_j \ll J^{3/4} p^{1/8} (\log p)^{1/4}.$$

*Proof.* We have

$$\sum_{j=1}^J (-1)^{j-1} d_j = \sum_{n=0}^{n_J-1} \left( \frac{n!}{p} \right)$$

From the *Polya-Vinogradov bound*

$$\max_{\chi \in \mathcal{X}^*} \max_{0 \leq h \leq k \leq p-1} \left| \sum_{c=h+1}^k \chi(c) \right| \ll p^{1/2} \log p \quad (4)$$

we see that  $n_J = 2J + O(p^{1/2} \log p) \ll J$  and by Theorem 1 we derive the result.  $\square$

Obviously

$$\sum_{j=1}^J d_j = n_J = 2J + O(p^{1/2} \log p),$$

which demonstrates that for every  $J \geq p^{1/2+\varepsilon}$  the odd and even spacings  $d_j$ ,  $j = 1, \dots, J$ , are of approximately the same total length.

We now denote by  $Q(H, N)$  the number of  $n = H + 1, \dots, H + N$  such that  $n!$  is a primitive root modulo  $p$ .

**Corollary 3.** *Let  $H$  and  $N$  be integers with  $0 \leq H < H + N < p$ . Then, for any fixed  $\varepsilon > 0$ , the following bound holds:*

$$Q(H, N) = N \frac{\varphi(p-1)}{p-1} + O(N^{3/4} p^{1/8+\varepsilon}).$$

**Theorem 4.** *Let  $H$  and  $N$  be integers with  $0 \leq H < H + N < p$ . Then for any fixed integer  $d \geq 0$ , the following bound holds:*

$$\max_{\deg f=d} \sum_{\chi \in \mathcal{X}} |T(\chi, f, H, N)|^2 \ll pN^{3/2}.$$

*Proof.* Arguing as in the proof of Theorem 1, and applying the Hölder inequality to (3), we derive that for any  $K$

$$\begin{aligned} |T(\chi, f, H, N)|^2 &\ll K^{-2} N \sum_{k,m=0}^{K-1} \sum_{n=H+1}^{H+N} {}^* \chi(\Psi_{k,m}(n)) \\ &\quad \mathbf{e}(f(n+k) - f(n+m)) + K^2, \end{aligned}$$

where

$$\Psi_{km}(X) = \prod_{i=1}^k (X+i) \prod_{j=1}^m (X+j)^{-1}$$

and as before  $\sum {}^*$  means that the poles of the corresponding rational function are excluded from the summation. Therefore,

$$\begin{aligned} \sum_{\chi \in \mathcal{X}} |T(\chi, f, H, N)|^2 \\ \ll K^{-2} N \sum_{k,m=0}^{K-1} \sum_{n=H+1}^{H+N} {}^* \sum_{\chi \in \mathcal{X}} \chi(\Psi_{k,m}(n)) + pK^2. \end{aligned}$$

The sum over  $\chi$  vanishes, unless

$$\Psi_{k,m}(n) \equiv 1 \pmod{p} \quad (5)$$

in which case it is equal to  $p - 1$ . For  $K$  pair  $(k, m)$  with  $k = m$  then there are  $N$  possible solutions to (5), for other  $O(K^2)$  pairs there are at most  $K$  solutions to (5). Thus

$$\begin{aligned} \sum_{\chi \in \mathcal{X}} |T(\chi, f, H, N)|^2 &\ll K^{-2}N (K^3 + KN) p + pK^2 \\ &= (NK + N^2K^{-1} + K^2) p. \end{aligned}$$

Taking  $K = \lfloor N^{1/2} \rfloor$ , we finish the proof.  $\square$

**Theorem 5.** *Let  $H$  and  $N$  be integers with  $0 \leq H < H + N < p$ . Then for any fixed integer  $\ell \geq 1$ , the following bound holds:*

$$\sum_{a=0}^{p-1} |S(a, H, N)|^2 \ll pN^{3/2}.$$

*Proof.* Arguing as in the proof of Theorem 1, we derive

$$\sum_{a=0}^{p-1} |S(a, H, N)|^2 \ll K^{-2}N \sum_{k,m=0}^{K-1} \sum_{n=H+1}^{H+N} \sum_{a=0}^{p-1} \mathbf{e}(an! \Phi_{k,m}(n)) + pK^2,$$

where

$$\Phi_{k,m}(X) = \prod_{i=1}^{k_\nu} (X + i) - \prod_{j=1}^{k_\nu+s} (X + j).$$

The sum over  $a$  vanishes, unless

$$n! \Phi_{k,m}(n) \equiv 0 \pmod{p}, \quad (6)$$

in which case it is equal to  $p$ .

As before, we see that  $\Phi_{k,m}(X)$  is a nonconstant polynomial of degree  $O(K)$ , unless  $k = m$ . Because  $n! \not\equiv 0 \pmod{p}$  for  $0 \leq H < n \leq H + N < p$ , we derive

$$\sum_{a=0}^{p-1} |S(a, H, N)|^2 \ll (NK + N^2K^{-1} + K^2) p.$$

Taking  $K = \lfloor N^{1/2} \rfloor$  and remarking that with this value of  $K$  the last term never dominates, we finish the proof.  $\square$



### 3 Sums and Products of Factorials

For integer  $\ell \geq 1$  and  $H$  and  $N$  with  $0 \leq H < H + N < p$  let us denote by  $I_\ell(H, N)$  and  $J_\ell(H, N)$  the number of solutions to the congruences

$$\prod_{i=1}^{\ell} n_i! \equiv \prod_{i=\ell+1}^{2\ell} n_i! \pmod{p}, \quad H+1 \leq n_1, \dots, n_{2\ell} \leq H+N,$$

and

$$\sum_{i=1}^{\ell} n_i! \equiv \sum_{i=\ell+1}^{2\ell} n_i! \pmod{p}, \quad H+1 \leq n_1, \dots, n_{2\ell} \leq H+N,$$

respectively.

From the properties of multiplicative and additive characters we immediately conclude that

$$\frac{1}{p-1} \sum_{\chi \in \mathcal{X}} |T(\chi, f, H, N)|^{2\ell} \leq \frac{1}{p-1} \sum_{\chi \in \mathcal{X}} |T(\chi, H, N)|^{2\ell} = I_\ell(H, N) \quad (7)$$

and

$$\frac{1}{p} \sum_{a=0}^{p-1} |S(a, H, N)|^{2\ell} = J_\ell(H, N). \quad (8)$$

The same arguments as in the proof of Theorem 5 lead to the bound

$$J_\ell(H, N) \ll N^{2\ell-1+1/(\ell+1)}.$$

We now show that for  $I_\ell(H, N)$  one can derive a more precise estimate.

**Theorem 6.** *Let  $H$  and  $N$  be integers with  $0 \leq H < H + N < p$ . Then for any fixed integer  $\ell \geq 1$ , the following bound holds:*

$$I_\ell(H, N) \ll N^{2\ell-1+2^{-\ell}}.$$

*Proof.* We prove this bound by induction. If  $\ell = 1$  then Theorem 4 taken with  $f(X) = 0$ , together with (7) immediately imply the desired bound  $I_1(H, N) \ll N^{3/2}$ .

Now assume that  $\ell \geq 2$  and that  $I_{\ell-1}(H, N) \ll N^{2\ell-3+2^{-\ell+1}}$ . We fix some  $K < N$  and note that by the Cauchy inequality we have

$$\begin{aligned} \left| \sum_{n=H+1}^{H+N} \chi(n!) \right|^2 &= \left| \sum_{k=1}^K \sum_{H+(k-1)N/K < m \leq H+kN/K} \chi(n!) \right|^2 \\ &\leq K \sum_{k=1}^K \left| \sum_{H+(k-1)N/K < m \leq H+kN/K} \chi(n!) \right|^2. \end{aligned}$$

Therefore

$$\begin{aligned} I_\ell(H, N) &= \frac{K}{p-1} \sum_{k=1}^K \sum_{\chi \in \mathcal{X}} \left| \sum_{H+(k-1)N/K < m \leq H+kN/K} \chi(n!) \right|^2 \left| \sum_{n=H+1}^{H+N} \chi(n!) \right|^{2\ell-2} \\ &= K \tilde{I}_\ell(K, H, N), \end{aligned}$$

where  $\tilde{I}_\ell(K, H, N)$  is the number of solutions to the congruence

$$m_1! \prod_{i=1}^{\ell-1} n_i! \equiv m_2! \prod_{i=\ell}^{2\ell-2} n_i! \pmod{p}$$

with  $H+1 \leq n_1, \dots, n_{2\ell-2} \leq H+N$  and  $H+(k-1)N/K < m_1, m_2 \leq H+kN/K$  for some  $k = 1, \dots, K$ . For each of  $N$  pairs  $(m_1, m_2)$  with  $m_1 = m_2$ , there are exactly  $I_{\ell-1}(H, N)$  solutions. Also we see that if  $n_1, \dots, n_{2\ell-2}$  are given then for each fixed value of  $r = m_1 - m_2$ , there are no more than  $|r|$  values solutions in  $m_1, m_2$  (because at least one of  $m_1, m_2$  satisfies a nontrivial polynomial congruence of degree  $|r|$ ). Certainly  $r = O(N/H)$ . Putting everything together and using the induction assumption we obtain

$$\tilde{I}_\ell(K, H, N) \ll N I_{\ell-1}(H, N) + (N/K)^2 N^{2\ell-2} = N^{2\ell-2+2^{-\ell+1}} + N^{2\ell} K^{-2}.$$

Therefore  $I_\ell(H, N) \ll K N^{2\ell-2+2^{-\ell+1}} + N^{2\ell} K^{-1}$ . Choosing  $K = \left\lceil N^{1-2^{-\ell}} \right\rceil$ , we obtain the desired bound.  $\square$

We now show that, for  $N \geq p^{1/2+\varepsilon}$  the above bound on  $I_\ell(H, N)$ , combined with Theorem 1, produces an asymptotic formula for  $I_\ell(H, N)$ . In particular for  $H = 0$ ,  $N = p - 1$ , this asymptotic formula is nontrivial for  $\ell \geq 4$ .

**Theorem 7.** *Let  $H$  and  $N$  be integers with  $0 \leq H < H + N < p$ . Then for any fixed integers  $\ell \geq r \geq 1$ , the following bound holds:*

$$I_\ell(H, N) = \frac{N^{2\ell}}{p-1} + O\left(N^{3\ell/2+r/2-1+2^{-r}} p^{(\ell-r)/4} (\log p)^{(\ell-r)/2}\right).$$

*Proof.* Similar to [8], we have

$$\begin{aligned} I_\ell(H, N) &= \frac{1}{p-1} \sum_{\chi \in \mathcal{X}} \left| \sum_{n=H+1}^{H+N} \chi(n!) \right|^{2\ell} \\ &= \frac{N^{2\ell}}{p-1} + \frac{1}{p-1} \sum_{\chi \in \mathcal{X}^*} \left| \sum_{n=H+1}^{H+N} \chi(n!) \right|^{2\ell} \\ &= \frac{N^{2\ell}}{p-1} + O\left( \frac{1}{p-1} \max_{\chi \in \mathcal{X}^*} \left| \sum_{n=H+1}^{H+N} \chi(n!) \right|^{2\ell-2r} \sum_{\chi \in \mathcal{X}} \left| \sum_{n=H+1}^{H+N} \chi(n!) \right|^{2r} \right), \end{aligned}$$

(note that in the last sum we bring back the term corresponding to  $\chi = \chi_0$ ). The result follows from (7) and Theorems 1 and 6.  $\square$

In particular, using Theorem 7 with  $r = 1$  we obtain

$$I_4(0, p-1) = p^7 \left(1 + O(p^{-1/4} (\log p)^{3/2})\right).$$

We now denote by  $F_\ell(a, H, N)$  the number of solutions to the congruence

$$\prod_{i=1}^{\ell} n_i! \equiv a \pmod{p}, \quad H+1 \leq n_1, \dots, n_\ell \leq H+N,$$

where  $a \in \mathbb{F}_p^*$ .

The same arguments as the ones used in the proof of Theorem 7 imply:

**Theorem 8.** *Let  $H$  and  $N$  be integers with  $0 \leq H < H + N < p$ . Then, for any fixed integers  $\ell \geq 2r \geq 1$ , the following bound holds:*

$$\max_{a \in \mathbb{F}_p^*} \left| F_\ell(a, H, N) - \frac{N^\ell}{p-1} \right| \ll N^{3\ell/4+r/2-1+2^{-r}} p^{(\ell-2r)/8} (\log p)^{(\ell-2r)/4}.$$

In particular, using Theorem 8 with  $r = 1$  we obtain

$$F_7(a, 0, p-1) = p^6(1 + O(p^{-1/8}(\log p)^{5/4})),$$

and for any  $\varepsilon > 0$ , using Theorem 8 with  $r = 2$  we obtain

$$F_7(a, H, N) = \frac{N^7}{p}(1 + o(1)), \quad \text{for } N \geq p^{11/12+\varepsilon},$$

hold for all  $a \in \mathbb{F}_p^*$ .

Let  $V_\ell(H, N)$  be the number of  $a \in \mathbb{F}_p^*$  for which  $F_\ell(a, H, N) > 0$ , that is,

$$V_\ell(H, N) = \# \left\{ \prod_{i=1}^{\ell} n_i! \pmod{p}, \mid H+1 \leq n_1, \dots, n_\ell \leq H+N \right\}.$$

**Theorem 9.** *Let  $H$  and  $N$  be integers with  $0 \leq H < H+N < p$ . Then for any fixed integers  $\ell \geq r \geq 1$ , the following bound holds:*

$$V_\ell(H, N) = p + O\left(N^{-\ell/2+r/2-1+2^{-r}} p^{(\ell-r+8)/4} (\log p)^{(\ell-r)/2}\right).$$

*Proof.* We may assume that  $\ell \geq 2$ , otherwise there is nothing to prove. Let

$$\mathcal{E} = \left\{ h \in \mathbb{F}_p \mid h \not\equiv \prod_{i=1}^{\ell} n_i! \pmod{p}, \ H+1 \leq n_1, \dots, n_\ell \leq H+N \right\}.$$

Then,

$$\frac{1}{p-1} \sum_{\chi \in \mathcal{X}} \sum_{n_1, \dots, n_\ell = H+1}^{H+N} \sum_{h \in \mathcal{E}} \chi(n_1! \dots n_\ell! h^{-1}) = 0.$$

Separating the term corresponding to  $\chi_0$  and, for  $\chi \in \mathcal{X}^*$ , applying Theorem 1 to the sums over  $n_1, \dots, n_{\ell-r}$ , we obtain

$$\frac{\#\mathcal{E} N^\ell}{p-1} \leq \left( N^{3/4} p^{1/8} \log^{1/4} p \right)^{\ell-r} \frac{1}{p-1} \sum_{\chi \in \mathcal{X}^*} \left| \sum_{h \in \mathcal{E}} \chi(h) \right| \left| \sum_{n=H+1}^{H+N} \chi(n!) \right|^r.$$

As before, we now extend summation over all characters  $\chi \in \mathcal{X}$  and by the Cauchy inequality, we derive from (7) and Theorem 6

$$\begin{aligned} \left( \sum_{\chi \in \mathcal{X}} \left| \sum_{h \in \mathcal{E}} \chi(h) \right| \left| \sum_{n=H+1}^{H+N} \chi(n!) \right|^r \right)^2 &\leq \sum_{\chi \in \mathcal{X}} \left| \sum_{h \in \mathcal{E}} \chi(h^{-1}) \right|^2 \sum_{\chi \in \mathcal{X}} \left| \sum_{n=H+1}^{H+N} \chi(n!) \right|^{2r} \\ &= (p-1) I_r(H, N) \#\mathcal{E} \ll (p-1)^2 N^{2r-1+2^{-r}} \#\mathcal{E}. \end{aligned}$$

Therefore,

$$\frac{N^\ell \#\mathcal{E}}{p-1} \leq \left( N^{3/4} p^{1/8} \log^{1/4} p \right)^{\ell-r} \cdot \left( \#\mathcal{E} N^{2r-1+2^{-r}} \right)^{1/2},$$

which finishes the proof.  $\square$

In particular, using Theorem 9 with  $r = 2$  we see that for  $N > p^{6/7+\varepsilon}$ , we have that only  $o(p)$  residue classes modulo  $p$  cannot be represented as  $n_1!n_2!n_3!n_4! \pmod{p}$  with  $H+1 \leq n_1, n_2, n_3, n_4 \leq H+N$ .

We recall that the *discrepancy*  $D$  of a sequence of  $M$  points  $(\gamma_j)_{j=1}^M$  of the unit interval  $[0, 1]$  is defined as

$$D = \sup_{\mathcal{I}} \left| \frac{A(\mathcal{I})}{M} - |\mathcal{I}| \right|,$$

where the supremum is taken over the interval  $\mathcal{I} = [\alpha, \beta] \subseteq [0, 1]$  of length  $|\mathcal{I}| = \beta - \alpha$  and  $A(\mathcal{I})$  is the number of points of this set which belong to  $\mathcal{I}$  (see [4, 10]).

For an integer  $a$  with  $\gcd(a, p) = 1$ , we denote by  $D_\ell(a, H, N)$  the discrepancy of the sequence of fractional parts

$$\left\{ \frac{a}{p} \prod_{i=1}^{\ell} n_i! \right\}, \quad H+1 \leq n_1, \dots, n_\ell \leq H+N.$$

Obviously,

$$D_\ell(a, H, N) = \max_{0 \leq h \leq k \leq p-1} \left| \frac{1}{N^\ell} \sum_{c=h+1}^k F_\ell(a^{-1}c, H, N) - \frac{k-h}{p} \right| + O(p^{-1}), \quad (9)$$

thus Theorem 8 can be used to estimate  $D_\ell(a, H, N)$ . However, we show that the Polya–Vinogradov bound (4) leads to stronger results.

**Theorem 10.** *Let  $H$  and  $N$  be integers with  $0 \leq H < H+N < p$ . Then for any fixed integers  $\ell \geq 2r \geq 1$ , the following bound holds:*

$$\max_{1 \leq a \leq p-1} |D_\ell(a, H, N)| \ll N^{-\ell/4+r/2-1+2^{-r}} p^{(\ell-2r+4)/8} (\log p)^{(\ell-2r+4)/4}.$$

*Proof.* We have

$$\begin{aligned}
\sum_{c=h+1}^k F_\ell(a^{-1}c, H, N) - \frac{k-h}{p} N^\ell \\
= \frac{1}{(p-1)} \sum_{\chi \in \mathcal{X}^*} \sum_{c=h+1}^k \sum_{n_1, \dots, n_\ell=H+1}^{H+N} \chi \left( ac^{-1} \prod_{i=1}^{\ell} n_i! \right) \\
= \frac{1}{(p-1)} \sum_{\chi \in \mathcal{X}^*} \chi(a) \sum_{a=h+1}^k \bar{\chi}(c) \left( \sum_{n=H+1}^{H+N} \chi(n!) \right)^\ell.
\end{aligned}$$

Thus, applying the bound (4), we deduce

$$\begin{aligned}
\left| \sum_{c=h+1}^k F_\ell(a^{-1}c, H, N) - \frac{k-h}{p} N^\ell \right| \\
\ll \max_{\chi \in \mathcal{X}^*} \left| \sum_{n=H+1}^{H+N} \chi(n!) \right|^{\ell-2r} \sum_{\chi \in \mathcal{X}} \left| \sum_{n=H+1}^{H+N} \chi(n!) \right|^{2r} p^{-1/2} \log p,
\end{aligned}$$

and the result follows from (7) and Theorems 1 and 6.  $\square$

In particular, using Theorem 10 with  $r = 1$  we obtain that

$$\max_{1 \leq a \leq p-1} |D_3(a, 0, p-1)| = O(p^{-1/8} (\log p)^{5/4}),$$

and also that for any  $\varepsilon > 0$ ,

$$\max_{1 \leq a \leq p-1} |D_3(0, p-1)| = o(1), \quad \text{for } N \geq p^{5/6+\varepsilon}.$$

We also note that Theorem 10 implies that

$$\max_{1 \leq a \leq p-1} \left| \sum_{n=H+1}^{H+N} \mathbf{e} \left( a \prod_{i=1}^{\ell} n_i! \right) \right| \ll N^{-\ell/4+r/2-1+2^{-r}} p^{(\ell-2r+4)/8} (\log p)^{(\ell-2r+4)/4}$$

for  $\ell \geq 2r \geq 1$ .

Let  $G_\ell(a, N)$  be the number of solutions to the congruence:

$$\prod_{i=1}^{\ell} n_i! \equiv a \pmod{p},$$

in positive integers  $n_1, \dots, n_\ell$  with

$$\sum_{i=1}^{\ell} n_i = N.$$

It has been shown in [16] that for any  $\varepsilon$  and sufficiently large  $p$ ,  $G_\ell(a, N) > 0$  provided that  $\ell \geq p^\varepsilon$  and  $N - \ell > p^{1/2+\varepsilon}$ . In [6], the same result has been obtained under a much weaker condition  $N - \ell > p^{1/4+\varepsilon}$ . Here, concentrate on the value of  $\ell$  and show that it can be taken as  $\ell = O(1)$  provided  $N > p^{1/2+\varepsilon}$ .

**Theorem 11.** *For any fixed integer  $\ell \geq 1$  and any integer  $N$  with  $1 \leq N < p/\ell$ , the following bound holds:*

$$\max_{a \in \mathbb{F}_p^*} \left| G_\ell(a, N) - \frac{1}{p-1} \binom{N-1}{\ell-1} \right| \ll N^{3\ell/4} p^{(\ell+6)/8} (\log p)^{(\ell-2)/4}.$$

*Proof.* For  $a \not\equiv 0 \pmod{p}$ , we have

$$G_\ell(a, N) = \frac{1}{p-1} \sum_{\substack{n_1, \dots, n_\ell \geq 1 \\ n_1 + \dots + n_\ell = N}} \sum_{\chi \in \mathcal{X}} \chi \left( a^{-1} \prod_{i=1}^{\ell} n_i! \right),$$

where the sum is taken over all multiplicative characters  $\chi$  modulo  $p$ . Separating the contribution from the principal character  $\chi_0$ , we obtain

$$\left| G_\ell(a, N) - \frac{1}{p-1} \binom{N-1}{\ell-1} \right| \leq \frac{1}{p-1} R,$$

where

$$\begin{aligned} R &= \sum_{\chi \in \mathcal{X}^*} \chi(a^{-1}) \sum_{\substack{n_1, \dots, n_\ell \geq 1 \\ n_1 + \dots + n_\ell = s}} \chi \left( \prod_{i=1}^{\ell} n_i! \right) \\ &= \sum_{\chi \in \mathcal{X}^*} \chi(a^{-1}) \sum_{n_1, \dots, n_\ell = 1}^{\ell} \chi \left( \prod_{i=1}^{\ell} n_i! \right) \frac{1}{p} \sum_{c=0}^{p-1} \mathbf{e}(c(n_1 + \dots + n_\ell - N)), \end{aligned}$$

because if  $\ell N < p$  then the congruence  $n_1 + \dots + n_\ell \equiv s \pmod{p}$  with  $1 \leq n_1, \dots, n_\ell \leq N$  is equivalent to the equation  $n_1 + \dots + n_\ell = s$ . Therefore,

$$R \leq \frac{1}{p} \sum_{c=0}^{p-1} \sum_{\chi \in \mathcal{X}^*} \left| \sum_{n=1}^{\ell} \chi(n!) \mathbf{e}(cn) \right|^\ell.$$

Arguing as in the proof of Theorem 7, we derive the result follows from (7) and Theorems 1 and 6.  $\square$

For example, for any fixed  $\varepsilon > 0$  and  $p/\ell > N \geq p^{1/2+\varepsilon}$ , we have

$$G_\ell(a, N) = \frac{1}{p-1} \binom{N-1}{\ell-1} (1 + o(1))$$

for every fixed  $\ell > \varepsilon^{-1} + 4$ .

We remark that one can easily drop the condition  $N < p/\ell$  in Theorem 11.

Let  $F(a, H, N) = F_1(a, H, N)$  be the number of solutions of the congruence  $n! \equiv a \pmod{p}$ ,  $H+1 \leq n \leq H+N$ .

**Theorem 12.** *Let  $H$  and  $N$  be integers with  $0 \leq H < H+N < p$ . Then following bound holds:*

$$\max_{a \in \mathbb{F}_p^*} F(a, H, N) \ll N^{2/3}.$$

*Proof.* Let  $K > 0$  be a parameter to be chosen later. Let

$$\mathcal{A} = \{H+1 \leq n \leq H+N \mid a \equiv n! \pmod{p}\} = \mathcal{A}_1 \cup \mathcal{A}_2,$$

where

$$\mathcal{A}_1 = \{n \in \mathcal{A} \mid |n-m| > K \text{ for all } m \neq n, m \in \mathcal{A}\} \quad \text{and} \quad \mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1.$$

It is clear that  $\#\mathcal{A}_1 \ll N/K$ . Assume now that  $n \in \mathcal{A}_2$ . Then there exists a nonzero integer  $k$  with  $|k| \leq K$  and such that  $n! \equiv (n+k)! \pmod{p}$ . For each  $k$ , the above relation leads to a polynomial congruence in  $n$  of degree  $|k|$  and therefore it has at most  $|k| \leq K$  solutions  $n$ . Summing up over all values of  $k$  with  $|k| \leq K$ , we get that  $\#\mathcal{A}_2 \ll K^2$ . Thus,

$$F(a, H, N) = \#\mathcal{A} = \#\mathcal{A}_1 + \#\mathcal{A}_2 \ll \frac{N}{K} + K^2,$$

and choosing  $K = \lfloor N^{1/3} \rfloor$  we get the desired inequality.  $\square$

We have seen that the Wilson theorem immediately implies the inequality  $V_2(0, p-1) \geq (p-1)/2$ . We now show that this bound can be slightly improved.



**Theorem 13.** *The following bound holds:*

$$V_2(0, p-1) \geq \frac{5}{8}p + O(p^{1/2} \log^2 p).$$

*Proof.* By (2), we see that if  $a \equiv b^{-1} \pmod{p}$ ,  $1 \leq b \leq p-1$  is odd then  $a \in V_2(0, p-1)$ .

By (1), we see that if  $a \equiv c^{-1}(c+1)^{-1} \pmod{p}$  with some even  $c = 2u$ ,  $1 \leq c \leq p-3$ , then  $a \in V_2(0, p-1)$  too. Thus each such  $c$  which corresponds to an even  $b = 2v$  in the above representation, contributes one new element to  $V_2(0, p-1)$ . It is also clear that no more than two distinct values of  $c$  can contribute the same element.

Therefore,  $V_2(0, p-1) \geq (p-1)/2 + W/2$ , where  $W$  is the number of solutions of the congruence

$$2u(2u+1) \equiv 2v \pmod{p}, \quad 0 \leq u, v \leq (p-3)/2.$$

The Weil bound yields  $W = p/4 + O(p^{1/2} \log^2 p)$  (see [1]), which concludes the proof.  $\square$

We remark that Theorem 13 immediately implies that for every integer  $a$  there exists a representation  $a \equiv n_1!n_2! + n_3!n_4! \pmod{p}$  with some positive integers  $n_1, n_2, n_3, n_4$ .

## 4 Concluding Remarks

Most of our results hold in more general settings. For example, let  $m \geq 1$  be any fixed positive integer and put

$$T(m, \chi, f, H, N) = \sum_{n=H+1}^{H+N} \chi \left( \prod_{\nu=1}^m (n + \nu - 1)! \right) \mathbf{e}(f(n)).$$

Then Theorem 1 holds with  $T(\chi, f, H, N)$  replaced by  $T(m, \chi, f, H, N)$ . In particular, if we write  $Q(m, H, N)$  for the number of  $n = H+1, \dots, H+N$  such that  $n!, \dots, (n+m-1)!$  are all primitive roots modulo  $p$ , then the estimate

$$Q(m, H, N) = N \left( \frac{\varphi(p-1)}{p-1} \right)^m + O \left( N^{1-1/2} p^{(\ell+2)/4\ell(\ell+1)} \right)$$

holds for any fixed integer  $\ell \geq 1$ .

Let  $\mathcal{Q}$  be the set of all distinct prime divisors of  $p - 1$ . For a set  $\mathcal{R} \subseteq \mathcal{Q}$ , we denote by  $T(\mathcal{R}, H, N)$  the number of  $n = H + 1, \dots, H + N$  such that for every  $q \in \mathcal{Q}$ ,  $n!$  is a  $q$ th power residue modulo  $p$  if and only if  $q \in \mathcal{R}$ . Then the estimate

$$T(\mathcal{R}, H, N) = N \prod_{q \in \mathcal{R}} \frac{q-1}{q} \prod_{q \in \mathcal{Q} \setminus \mathcal{R}} \frac{1}{q} + O(N^{1-1/2\ell} p^{(\ell+2)/4\ell(\ell+1)})$$

holds for any fixed integer  $\ell \geq 1$ .

Techniques of the present paper apply also to the sequences

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}, \quad (2n+1)!! = 1 \cdot 3 \cdot \dots \cdot (2n+1),$$

and many others, as well as their combinations.

Also, with some minor adjustments, our methods can be used to obtain similar, albeit somewhat weaker results for composite moduli. In this setup, our basic tools such as the Weil bound and the Lagrange theorem, have to be replaced with their analogues in residue rings modulo a composite number. See, for example, [3] for bounds of character sums, and [9] for bounds on the number of small solutions of polynomial congruences.

While the results of the present paper represent some progress towards better understanding the behaviour of  $n!$  modulo  $p$ , there are several challenging questions that deserve further investigation. For example, our Theorem 12 gives a nontrivial upper bound on  $F(a, H, N)$ , but we conjecture that this result is far from being sharp. We do not have any nontrivial individual upper bounds for  $S(a, H, N)$ .

Certainly, studying  $V_1(H, N)$  is of primal interest. Trivially, we have  $V_1(H, N) \geq (N-1)^{1/2}$  (to see this it is enough to recall that  $n = n!/(n-1)!$ ), but we have not been able to obtain any better lower bound. In the opposite direction, answering a question of Erdős, Rokowska and Schinzel [17] have showed that if the residues of  $2!, 3!, \dots, (p-1)!$  modulo  $p$  are all distinct, then the missing residue must be that of  $-((p-1)/2)!$ , that  $p \equiv 5 \pmod{8}$ , and that no such  $p$  exists in the interval  $[7, 1000]$ , but it does not seem to be even known that there can be only finitely many such  $p$ , or, equivalently, that  $V_1(0, p-1) = p-2$  can happen only for finitely many values of the prime  $p$ .

It is very tempting to try to generalize the proof of Theorem 13 and consider longer products  $c(c+1) \dots (c+m)$ . This may lead to an improvement of the constant  $5/8$  of Theorem 13. However, to implement this strategy one has to study in detail image sets of such polynomials (and their overlaps), which may involve rather complicated machinery.

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